The description of phase transition in a black hole with conformal anomaly in the Ehrenfest's scheme

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Abstract

We make use of the Ehrenfest's equation to explore the phase transition of a black hole with conformal anomaly. The first order phase transition is ruled out because no discontinuity appears in entropy of the black holes. We find that the phase transition of black holes belong to the second order subject to the Ehrenfest's equations. Further we also show that the second order phase transition will not happen for the black holes without conformal anomaly.

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I. Introduction

The Einstein-Gauss-Bonnet gravity including various spacetimes is of considerable interest motivated by developments in the string theory. The theory is a special case of Lovelock's theory of gravitation [1]. The Gauss-Bonnet term arising naturally in the low-energy limit of heterotic superstring theory is the first and dominating quantum correction to classical general relativity. This term appears as quadratic in the curvature of the spacetime in the Lagrangian. The Gauss-Bonnet term regularizes the metric and modifies the Friedmann equation. The estimation on the Gauss-Bonnet effect has been performed [2-8]. As an important concept in quantum field theory in curved spacetimes, the conformal anomaly is inserted into the energy-momentum tensor within the gravity and cosmology. The conformal anomaly hires the Gauss-Bonnet term. It is significant to probe its influence in different directions.

Recently a lot of attentions were paid to the thermodynamic quantities and phase transitions on various black holes due to the area law of the black hole entropy provoked by Bekenstein and Hawking [9-11]. The black holes can be thought as thermodynamic system. The works on phase transitions of black holes in the frame of semiclassical gravity were listed [12]. The thermodynamic characteristics of modified Schwarzschild black hole have been researched in the Ref. [7, 13]. The thermodynamic phase transition in Born-Infeld-anti de Sitter black holes were discussed in virtue of various ways [14, 15]. The phase transition of the quantum-corrected Schwarzschild black hole was investigated, which fosters the research on the quantum-mechanical aspects of thermodynamic behaviours [16]. The thermodynamic quantities of a black hole involving an $f(R)$ global monopole were also evaluated [17]. In the process of research on the phase transition of black holes, a new idea based on the Clausius-Clapeyron scheme or the Ehrenfest's scheme was put forward by Banerjee et. al. [18, 19]. Further the phase transition in Reissner-Nordstrom-AdS black holes and Kerr-AdS black holes were discussed with the help of Ehrenfest's equations [19-21].

In this paper we plan to study the phase transition of static and spherically symmetric black holes with conformal anomaly in the Ehrenfest's scheme classifying the phase transitions as the first order or higher order transition. The thermodynamic quantities and phase transition of black holes with Gauss-Bonnet corrections were discussed [7, 22]. It is significant to elaborate how the Gauss-Bonnet effect control the black holes phase transition. We should understand which kind of phase transition happens for the black hole while how their thermodynamic quantities behave. To our knowledge, little contribution is made to estimate the influence from Gauss-Bonnet term on the transition. We wish to make use of Ehrenfest's equations to describe the phase transition of the spherically symmetric black holes whose effective energy-momentum tensor contains the Gauss-Bonnet term. We follow the procedure of Ref. [18, 19] to study the evolution of this kind of black holes. First of all the exact soluble metric with the quantum black reaction will be introduced. We analyze the nature of the phase transition of the Gauss-Bonnet corrected black hole by means of Ehrenfest's scheme to show the relation between the phase transition and the Gauss-Bonnet term. We derive the thermodynamic variables such as the heat capacity etc. and discuss their

singularities at the critical point to exhibit which order the phase transition of the black hole is. Finally the arguments will be listed.

II. The thermodynamics of black hole with conformal anomaly

When the Gauss-Bonnet term is inserted into the effective energy-momentum tensor, the static and spherically symmetric metrics as solutions to the Einstein equation are found as,

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

where

$$f(r) = 1 - \frac{r^2}{4\alpha}(1 - \sqrt{1 - \frac{16\alpha M}{r^3} + \frac{8\alpha Q^2}{r^4}}) \tag{2}$$

and here M and Q are integration constants. The coefficient α is positive. We set the Newton constant $G_N = 1$. The asymptotic behaviour of this line element is given by,

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + O(\alpha, r^{-2}) \tag{3}$$

which is the form of Reissner-Nordstrom black holes with the mass M and the electric charge Q asymptotically. The outer horizon is a root of metric (2) like $f(r_+) = 0$, then the mass of the black hole can be written as,

$$M = \frac{r_+}{2} + \frac{Q^2 - 2\alpha}{2r_+} \tag{4}$$

The relation between mass M of a black hole with conformal anomaly and the outer horizon r_+ is plotted in Figure 1. When $r_+ = \sqrt{Q^2 - 2\alpha}$, the mass of black hole achieves the minimum. The mass function decreases while $0 < r_+ < \sqrt{Q^2 - 2\alpha}$. According to Ref. [7], the modified entropy is,

$$S = \pi r_+^2 - 4\pi\alpha \ln r_+^2 \tag{5}$$

The first law of black hole thermodynamics is,

$$dM = T_H dS + \Phi dQ \tag{6}$$

where T_H is the Hawking temperature denoted as,

$$\begin{aligned} T_H &= \frac{f'(r_+)}{4\pi} \\ &= \frac{1}{4\pi}\frac{r_+^2 + 2\alpha - Q^2}{r_+(r_+^2 - 4\alpha)} \end{aligned} \tag{7}$$

and Φ is the potential difference between the horizon and the infinity like,

$$\phi = \frac{Q}{r_+} \tag{8}$$

and the heat capacity at constant potential of the black hole is defined as,

$$\begin{aligned} C_\phi &= T_H \left(\frac{\partial S}{\partial T_H}\right)_\phi \\ &= -\frac{2\pi(r_+^2 - 4\alpha)^2(r_+^2 - Q^2 + 2\alpha)}{r_+^4 - (Q^2 - 10\alpha)r_+^2 - (4\alpha Q^2 + 8\alpha^2)} \end{aligned} \tag{9}$$

Having found the roots of the denominator of the heat capacity, we obtain the points of phase transition as follow,

$$r_c = \frac{\sqrt{2}}{2}\sqrt{Q^2 - 10\alpha + \sqrt{Q^4 - 4\alpha Q^2 + 132\alpha^2}} \tag{10}$$

If the horizon of black hole is equal to r_c, the phase transition will emerge. According to expression (9), the dependence of heat capacity on the horizon is shown graphically in Figure 2. At points of phase transition the heat capacity is not continuous while the divergence generates. It is clear that the heat capacity is discontinuous at $r_+ = r_c$ which is the phase transition point. At this point, C_ϕ flips from negative infinity to positive infinity. It indicates that black hole transform from unstable phase 1 where the heat capacity is negative into stable phase 2 where C_ϕ is positive at the critical point. In addition to Figure 1, because of the negative slope of $M(r_+)$ while taking the point close to phase transition point, the phase transition of a black hole with conformal anomaly can be described as an unstable black hole with larger mass in phase 1 turns to a stable black hole with smaller mass in phase 2. We also plot the case for Reissner-Nordstrom black hole in Figure 2. The relation between heat capacity and entropy for Reissner-Nordstrom black hole can be easily expressed as C_ϕ, which is completely different from the case for black hole with conformal anomaly, but it can be obtained from (9) by taking $\alpha = 0$. It is a continuous straight line, so no phase transition in this order for Reissner-Nordstrom black hole occurs. It can be checked that higher order derivation of C_ϕ for Reissner-Nordstrom black hole is constant, which means that this kind of black hole will not perform phase change in any order.

III. The description of a black hole with conformal anomaly in the Ehrenfest's scheme

In the standard thermodynamics, the phase transition is fundamental phenomena for a thermodynamical system. A phase transition means that a discontinuity of a state space variable occurs. That which kind of state variables have a discontinuity at the critical points determines which kind of phase transitions. We can think that the black holes are thermodynamic objects. The evolution of black holes can be classified some kinds of phase transition or not.

First of all we focus on the Hawking temperature (7) and the relationship between the temperature and entropy is plotted in Figure 3. We can see that the shape of these curves for black hole with conformal anomaly are similar to Reissner-Nordstrom black hole, especially when α is

small. The slope of temperature function is positive at the points adjacent to the critical point r_c. The temperature of a black hole with conformal anomaly in phase 1 is smaller than in phase 2. It is evident that the curves are continuous, which means that the evolution of the black holes with conformal anomaly does not belong to the first order phase transition.

Next we are going to check the existence of the second order phase transition describing how the black holes with conformal anomaly undergo. The first and second Ehrenfest's equations for black holes are,

$$-\frac{d\phi}{dT_H}=\frac{1}{T_H Q}\frac{C_{\phi 2}-C_{\phi 1}}{\beta_2-\beta_1} \tag{11}$$

$$-\frac{d\phi}{dT_H}=\frac{\beta_2-\beta_1}{\kappa_2-\kappa_1} \tag{12}$$

where the subscripts 1 and 2 stand for phase 1 and 2 respectively. The analog of volume expansion coefficient is denoted as,

$$\beta=\frac{1}{Q}(\frac{\partial Q}{\partial T_H})_\phi \tag{13}$$

Similarly, the analog of isothermal compressibility is defined as,

$$\kappa=\frac{1}{Q}(\frac{\partial Q}{\partial \phi})_{T_H} \tag{14}$$

Further we discuss the new variables β and κ by means of Eq. (7) and (8),

$$\beta=-\frac{4\pi r_+(r_+^2-4\alpha)^2}{r_+^4-(Q^2-10\alpha)r_+^2-4\alpha(Q^2+2\alpha)} \tag{15}$$

$$\kappa=\frac{r_+}{Q}\frac{r_+^4-(3Q^2-10\alpha)r_+^2+4\alpha(Q^2-2\alpha)}{r_+^4-(Q^2-10\alpha)r_+^2-4\alpha(Q^2+2\alpha)} \tag{16}$$

The new variables β correspond volume expansivity and κ correspond isothermal compressibility can be seen in Figure 4 and Figure 5 respectively. It is interesting that both the two thermodynamic quantities are discontinuous at the points of phase transition. The natures of variables C_Φ, β and κ predict that higher order phase transition in the black holes will emerge. For Reissner-Nordstrom black hole, volume expansivity β and isothermal compressibility κ are $\beta=-\frac{4\pi r_+^3}{r_+^2-Q^2}$, $\kappa=\frac{r_+}{Q}\frac{r_+^2-3Q^2}{r_+^2-Q^2}$, which are also plotted in Figure 4 and 5. They are divergent at $r_+=Q$. These two quantities diverse contrast to the behaviour of C_ϕ for Reissner-Nordstrom black hole.

Heat capacity C_ϕ can be presented as $C_\phi=\frac{B(r_+)}{A(r_+)}$, and $B(r_+)=-2\pi(r_+^2-4\alpha)^2(r_+^2-Q^2+2\alpha)$, $A(r_+)=r_+^4-(Q^2-10\alpha)r_+^2-(4\alpha Q^2+8\alpha^2)$, then

$$C_{\phi 2}-C_{\phi 1}=\frac{B(r_{+2})}{A(r_{+2})}-\frac{B(r_{+1})}{A(r_{+1})} \tag{17}$$

We choose two points close to the phase transition point, then $B(r_{+1})=B(r_{+2})=B(r_c)\neq 0$. This approximation is not satisfied with $A(r_+)$ since the denominator $A(r_c)=0$, therefore,

$$C_{\phi 2}-C_{\phi 1}=B(r_c)(\frac{1}{A(r_{+2})}-\frac{1}{A(r_{+1})}) \tag{18}$$

Apply the approximate method to β and κ, then

$$\beta_2-\beta_1=C(r_c)(\frac{1}{A(r_{+2})}-\frac{1}{A(r_{+1})}) \tag{19}$$

$$\kappa_2-\kappa_1=D(r_c)(\frac{1}{A(r_{+2})}-\frac{1}{A(r_{+1})}) \tag{20}$$

where $C(r_c)=-4\pi r_c(r_c^2-4\alpha)^2$, $D(r_c)=r_c(r_c^4-r_c^2(3Q^2-10\alpha)+4\alpha(Q^2-2\alpha))/Q$. Now we combine the Ehrenfest's equation (11), heat capacity (9) and the analog of volume expansion coefficient (13) at phase transition points (10) to obtain,

$$\frac{1}{TQ}\frac{C_{\phi 2}-C_{\phi 1}}{\beta_2-\beta_1}|_{r_+=r_c}=\frac{2\pi(r_c^2-4\alpha)}{Q} \tag{21}$$

Having taken into account the Ehrenfest's equation (12), the analog of volume expansion coefficient (13) and the analog of isothermal compressibility (14) at the critical points (10), we find

$$\frac{\beta_2-\beta_1}{\kappa_2-\kappa_1}|_{r_+=r_c}=\frac{2\pi(r_c^2-4\alpha)}{Q} \tag{22}$$

Although the variables C_Φ, β and κ are divergent at critical points, their divergence will be cancelled in the Ehrenfest's equations. According to Eq. (21) and (22), the condition is,

$$\frac{\triangle C_\phi}{TQ}=\frac{\triangle\beta^2}{\triangle\kappa} \tag{23}$$

where $\triangle C_\phi=C_{\phi 2}-C_{\phi 1}$, $\triangle\beta=\beta_2-\beta_1$ and $\triangle\kappa=\kappa_2-\kappa_1$. If the variables of evolving black holes satisfy the condition (23), the evolution is the second order phase transition. For Reissner-Nordstrom black hole, the heat capacity is $C_\phi=-2\pi r_+^2$ according to Eq. (9). From phase 1 to phase 2 or conversely, the difference of the heat capacities for the two kinds of phases respectively disappears $\triangle C_\Phi=0$, but $\triangle\beta\neq 0$ and $\triangle\kappa\neq 0$. The necessary condition (23) is violated. The second order phase transition of the black holes can not occur. It is interesting that the Gauss-Bonnet term is fundamental for the second order phase transition of the black holes. Only the black holes with conformal anomaly can undergo to perform the second order phase transition.

IV. Conclusion

In this work we discuss the phase transition in the black holes with conformal anomaly in the Ehrenfest's scheme and further show which kind of order the phase transition belongs to. The black holes are thought as thermodynamic objects and their thermodynamic quantities such as Hawking temperature, heat capacity at constant potential, the analog of volume expansion coefficient and isothermal compressibility are investigated from the first law of black hole thermodynamics. We find that the curves standing for the relationship between the Hawking temperature and the mass

of the black holes are continuous, so the first order phase transition is excluded in this description of evolution of black holes involving conformal anomaly. It is interesting that all of the heat capacity at constant potential, the variables correspond volume expansivity and isothermal compressibility are not continuous at phase transition points and these variables satisfy the Ehrenfest's equation. According to the plots and calculation, we find at the points close to the critical point $r_+ = r_c$, an unstable black hole with larger mass changes into a stable black hole with smaller mass. We prove analytically that the phase transition for the black holes is the second order. It is surprising that the heat capacity of black holes will recover to be continuous and finite at any points if the influence from conformal anomaly is omitted, then the second order phase transition will not emerge. For the black holes, the conformal anomaly induces the second order phase transition.

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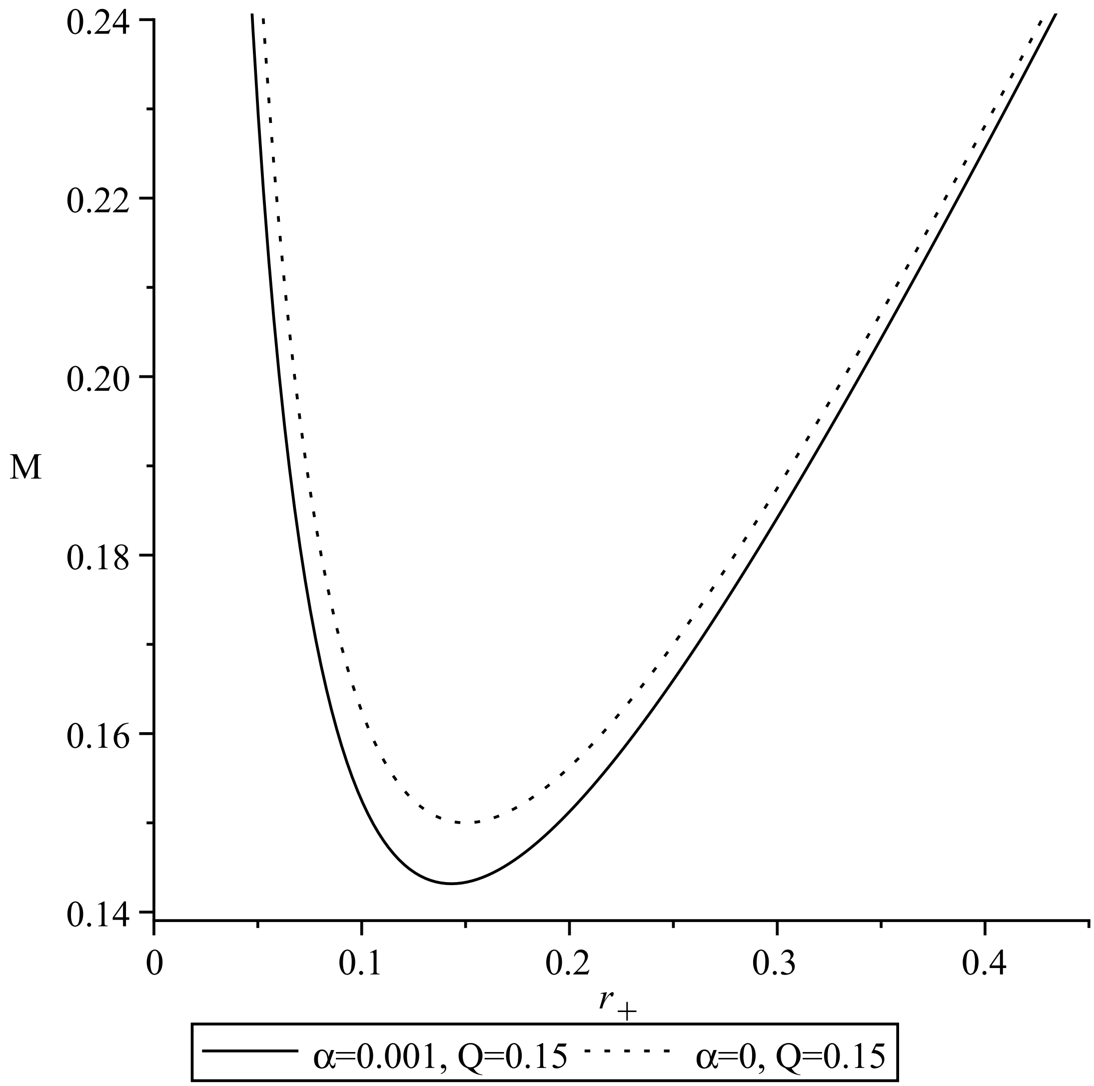

Figure 1: The behaviour of black hole mass $M(r_+)$ involving conformal anomaly and a Reissner-Nordstrom black hole with respect to the outer horizon of black hole.

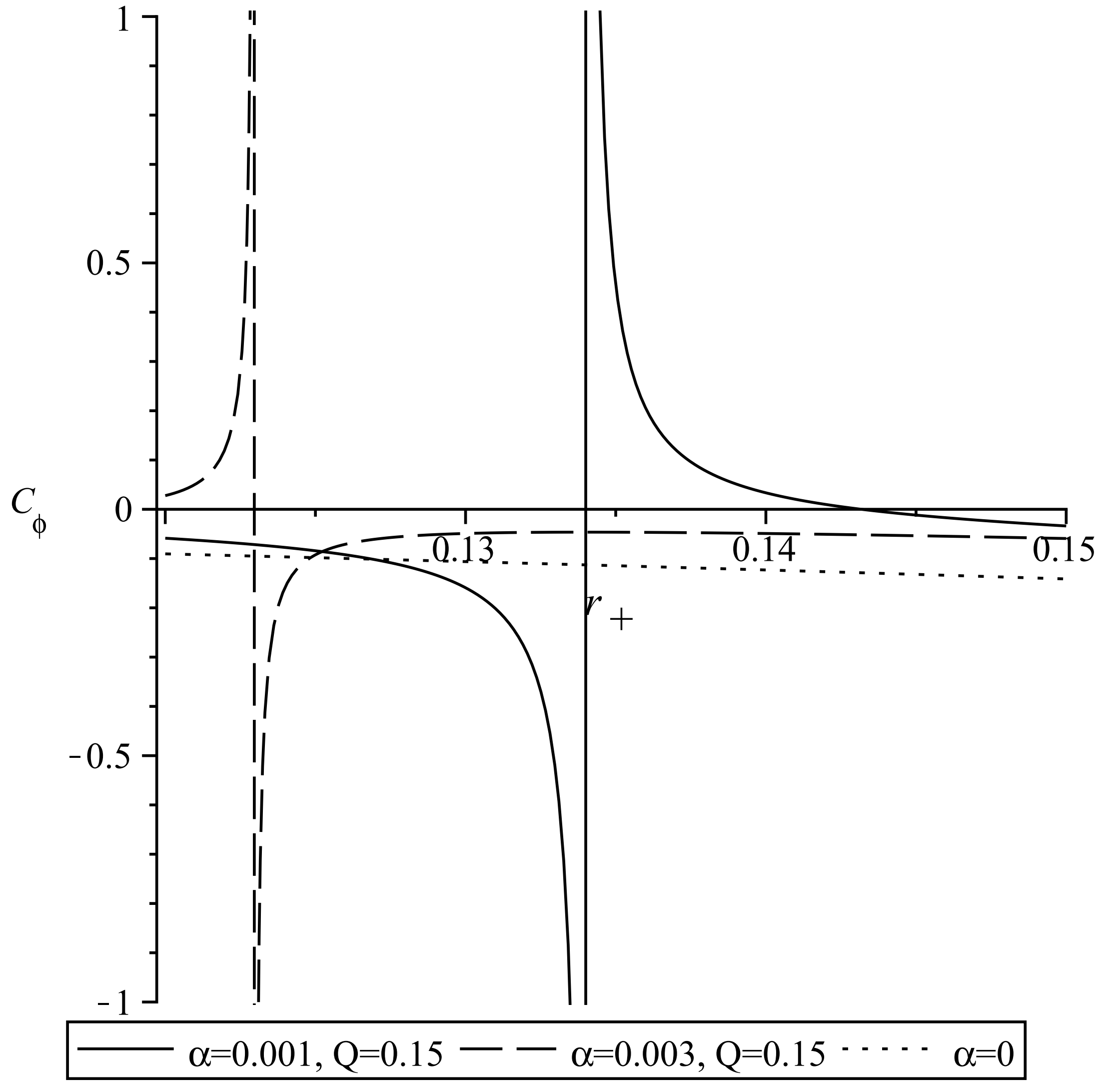

Figure 2: The solid line and dash line correspond the dependence of heat capacity of a black hole with conformal anomaly on the horizon of black hole with different α, and the dot line for Reissner-Nordstrom black hole.

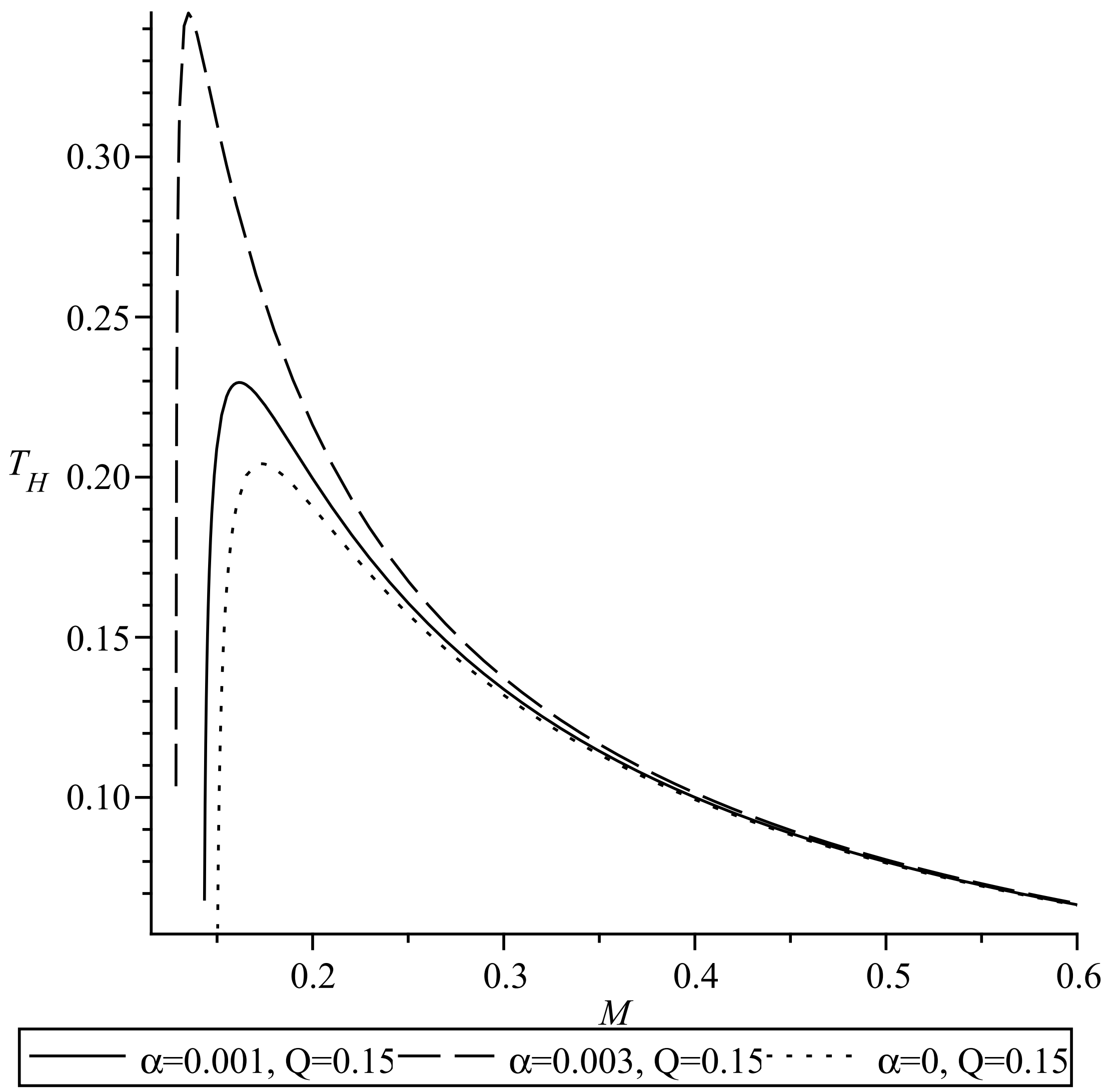

Figure 3: The relation between the Hawking temperature and the mass of a black hole with or without conformal anomaly.

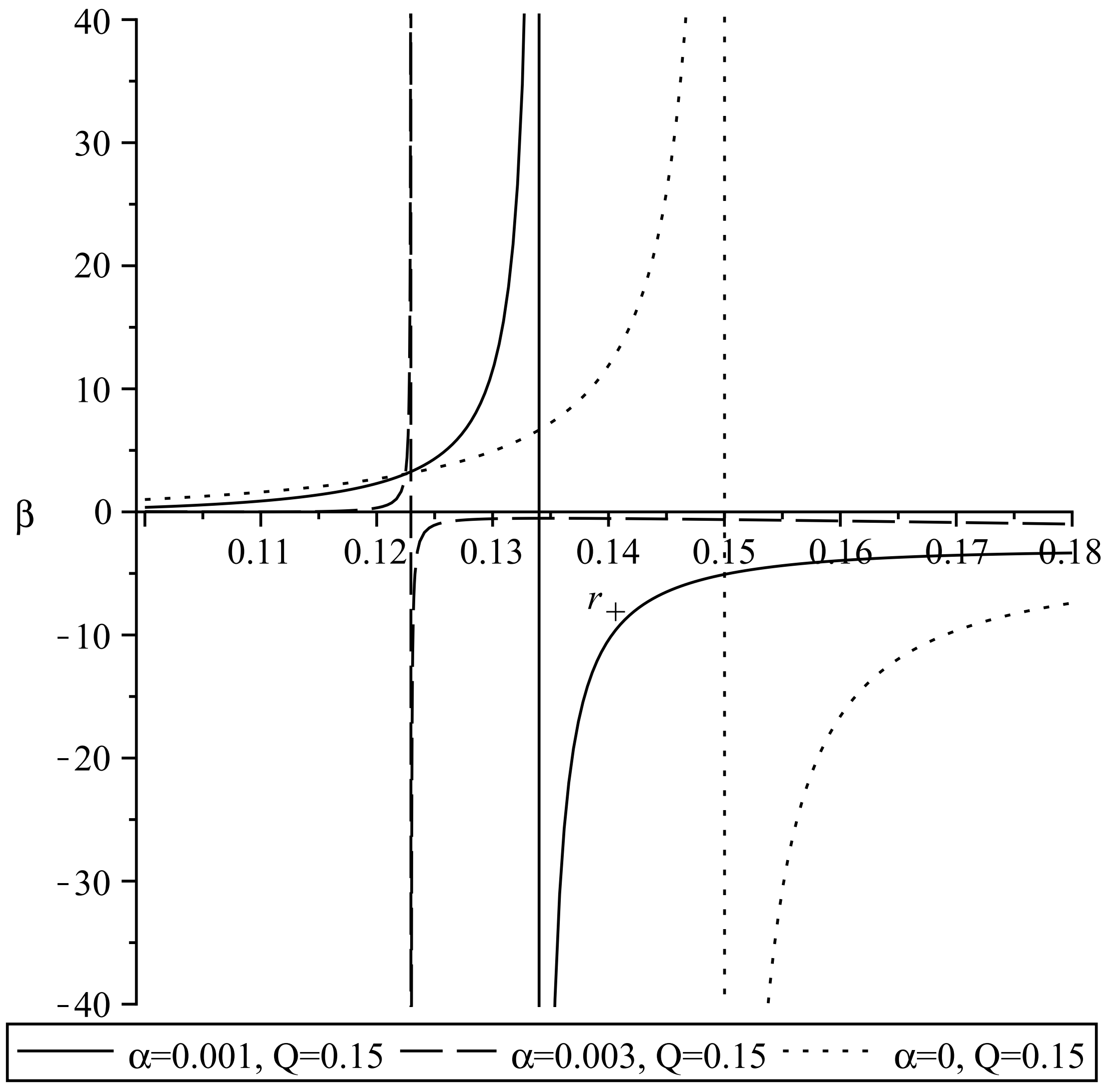

Figure 4: The dependence of the analog of volume expansion coefficient on the outer horizon for a black hole with different conformal anomaly α.

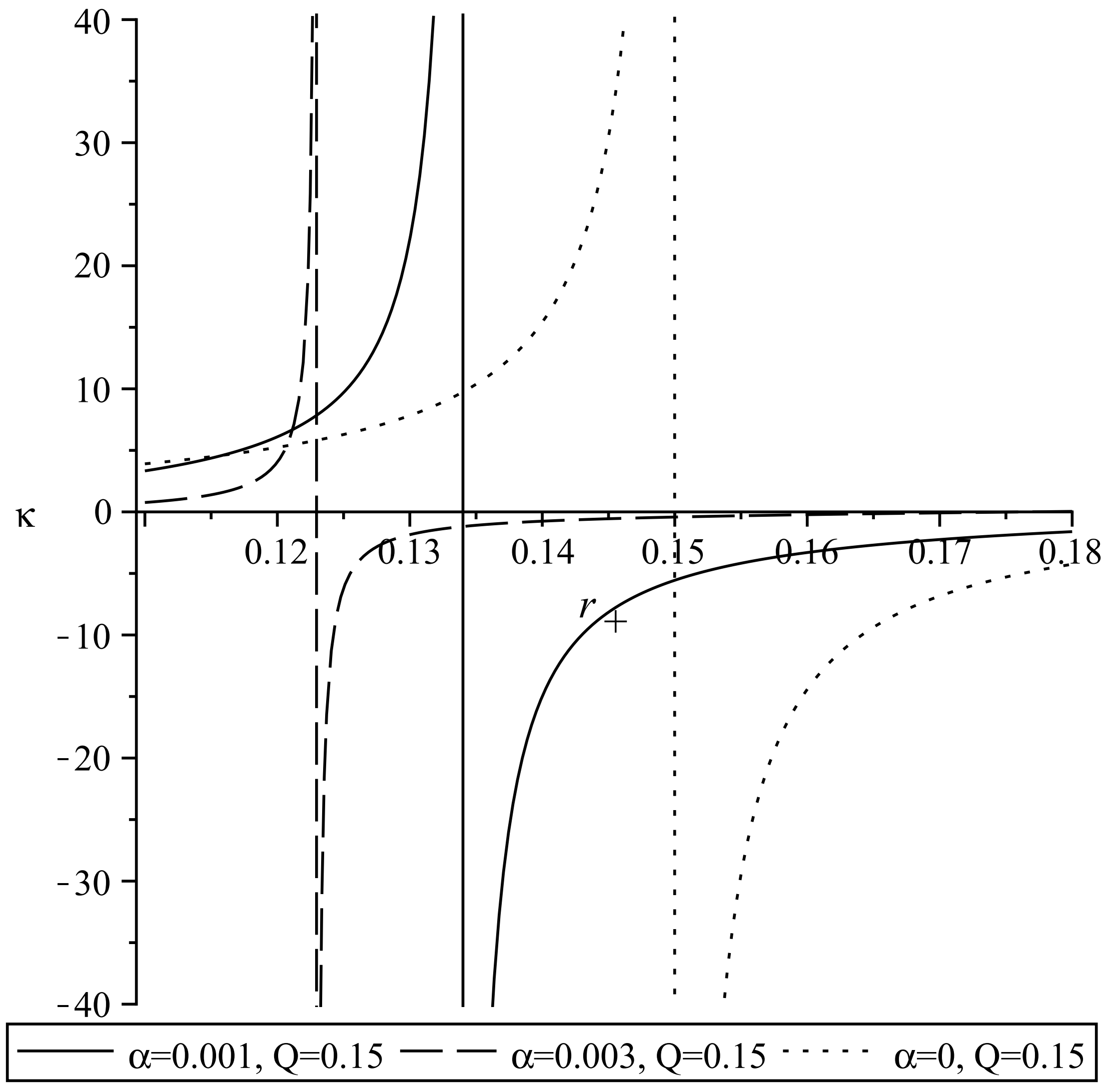

Figure 5: The dependence of the analog of isothermal compressibility on the outer horizon for a black hole with different conformal anomaly α.